Some syntactical and semantical properties for pair sentential calculus **PSC**

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Abstract

This paper is an extended version of my talk in the Conference of Non-Classical Logics 2016 [7]. In this paper we will introduce a system that rejects the principle of identity "A is A", one of the third Aristotelian principles for thinking. The proposed system allows to deal with paradoxical sentences, like a Liar sentence "A is not A". We present both an axiomatic system and an adequate semantics for it.

Keywords: **SCI**, Liar paradox, self-reference, revision theory, matrix semantics.

1 Introduction

In the 1970's, R. Suszko had attempted to formalize an ontology of facts in L. Wittgenstein's *Tractatus* on the basis of Fregean scheme, and called it *non-Fregean logic.* The sentential calculus with identity, **SCI** in short, is the most simplified version of his *non-Fregean logic* and can be obtained by adding the sentential identity connective $\equiv$ to the classical logic. Statements of the form $A \equiv B$ read as "A is identical with B", which means that the referent of two sentences are identical in the basis of Fregean scheme. From the axiom (SI): $(A \equiv B) \to (A \to B)$, the statement $A \leftrightarrow B$ obviously does not imply $A \equiv B$ and we may consider more than two situations (true and false), hence **SCI** is usually called a non-Fregean logic. Every equation in the logical theorems of **SCI** is only a trivial $A \equiv A$, so **SCI** is very weak but many logical systems can be simulated on Suszko's theories of situation [10].

We have paid attention to the simulation property of **SCI** and attempt to deal with a simple Liar sentence: "This sentence is not true" in **SCI**. Let's define A="This sentence is true", then we get $A \equiv \neg A$ because the referent of two sentences A and $\neg A$ are identical, but it's impossible logically by (SI). In order to overcome the matter, we have introduced a *referential relation* of pair-sentence similar to identity $\equiv$, i.e., $(A^0, \neg A^1)$, which means that a *situation* of A on stage 0 is *referential* to the *situation* of $\neg A$ on stage 1. The referential relation is similar to identity, but more general notion just as a mutual link relation between sentences, even that can be established between contradict sentences if we introduce the stage notion on which each sentence is valid. We had proposed a pair sentential calculus, **PSC** [6] in short, which was obtained from the classical one by adding a new pair-sentence connective $((\_)^i,\ (\_)^j)$, where i, j are some stage numbers.

It is usually assumed that several fundamental postulates implicitly hold in logical reasoning by a priori. These postulates are called the third Aristotelian principles for thinking. The first *principle of identity* says that "A is always A and not being $\neg A$", the second *principle of contradiction* says that "A is not both A and $\neg A$", and the third *principle of excluded middle* says that "either A is B or A is $\neg B$". If we reject some of them, we get several kinds of non-classical reasoning. For example [9], it is well known that de Morgan or intuitionistic reasonings are obtained from the classical one by rejecting both principles of second and third or principle

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of third only, respectively. But we also think that it is useful to reject the first principle of identity to proceed correctly the formal reasoning in several kinds of logical paradoxes that appear between definition and definiendum of sentences.

We interpret a set of pair-sentences $\{(A, B_0), (B_0, B_1), \ldots, (B_{n-1}, A)\}$ as a sequence of referential relation such that the referential recursive pattern: $A\ B_0\ B_1 \ldots B_{n-1} A\ B_0\ B_1\ \ldots$ holds by following the ideas of H. G. Herzberger [5], A. Gupta [4] and L. H. Kauffman [8]. Then for the principle of identity: "A is A", we get a pair-sentence (A^0, A^1) which satisfies a sequential form: $A\ A\ A\ A\ A\ A \ldots$. Similarly, for a simple Liar paradoxical sentence: "This sentence is not true", we get a pair-sentence $(A^0, (\neg A)^1)$ which satisfies a sequential form: $A\ \neg A\ A\ \neg A\ A\ \neg A \ldots$ under the assumption of $\neg\neg A \leftrightarrow A$ holds on each stage. We get typically four forms: (A^0, A^1), $(\neg A^0, \neg A^1)$, $(\neg A^0, A^1)$ and $(A^0, \neg A^1)$ as the most simple referential relations. (A^0, A^1) and $(\neg A^0, \neg A^1)$ correspond to truth-taller and false-taller, respectively, and both $(\neg A^0, A^1)$ and $(A^0, \neg A^1)$ to simple Liar paradoxes. If we will axiomatize the pair-sentence calculus similar to **SCI** manners, then the obtained system is not as one of four-valued logic [3], but as a classical two-valued logic according to Suszko's Thesis of bivalence [2].

2 PSC Logic

Let $\mathcal{L}_P =< \boldsymbol{FOR_P}, \neg, \wedge, \vee, \rightarrow, ((\_)^i,\ (\_)^j), \top, \bot >$ be a language of the sentential calculus with a pair-sentence connective. The formulas $\boldsymbol{FOR_P}$ of a language $\mathcal{L}_P$ are generated in the usual way from an infinite set $\boldsymbol{VAR_P}$ of sentential variables, constants $\top$(true) and $\bot$(false) by the standard truth functional connectives $\neg$(negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (material implication) as well as the pair-sentence constructor $((\_)^i,\ (\_)^j)$, where $i, j \in \mathbf{N}$ are some stage numbers. In our language $\mathcal{L}_P$, we assume that every sentential variables are defined on an initial stage number $0 \in \mathbf{N}$. So, we have:
(1) $\boldsymbol{VAR_P} = \bigcup_{i \in \mathbf{N}} \boldsymbol{VAR}^i$, where $\boldsymbol{VAR}^i = \{p^i, q^i, r^i, \ldots\}$ $(\forall i \in \mathbf{N})$
(2) $\boldsymbol{VAR_P} \subseteq \boldsymbol{FOR_P}$
(3) $\forall A, B \in \boldsymbol{FOR_P} \Longrightarrow \neg A, A \wedge B, A \vee B, A \rightarrow B, (A, B) \in \boldsymbol{FOR_P}$

Also we may use the same parentheses as auxiliary symbols even assume that the priority of each connective are weak as $\neg$, $\wedge$, $\vee$, $\rightarrow$, $(\_\,,\_)$ in order. Throughout this paper the letters p, q, r, p^0, p^1, p^2, $\ldots$ will be used to denote any variables, the letters A, B, C, A^0, A^1, A^2, $\ldots$ formulas of a language $\mathcal{L}_P$, the letters X, Y, Z, $\ldots$ sets of formulas, and Greek letters $\Gamma, \Sigma, \Delta, \ldots$ sets of pair-sentence formulas. Moreover, two constants $\top$ and $\bot$ are defined as $A^0 \vee (\neg A)^0$ and $A^0 \wedge (\neg A)^0$, respectively. At first we will introduce several terminology with pair-sentence as the following.

Definition 2.1 (Pair-sentence) (1) *For any sentence $A^0 \in \boldsymbol{FOR_P}$, if there exist some sentence $B^0 \in \boldsymbol{FOR_P}$ such that "A^0 is B^0" is also a new sentence, then we assume that there exists $(A^0, B^1) \in \boldsymbol{FOR_P}$, which means that there exists A^1 on the next stage of A^0 such that A^1 is referential to B^0, and call (A^0, B^1) a* pair-sentence formula *of A^0 and B^0. Otherwise, we assume that there exists a senetnce $(A^0, A^0) \in \boldsymbol{FOR_P}$, and call (A^0, A^0) a* unit of pair-sentence formula *for A^0. The superscript of each formula shows the referential stage number on which the formula is valid.*

(2) *The referential stage numbering of composed formulas is the following: for any stage numbers $i, j, k \in \mathbf{N}$,*
(i) $(\neg A^i)^j \iff \neg(A^{i+j})$
(ii) $(A^i \,\%\, B^j)^k \iff A^{i+k} \,\%\, B^{j+k}$ *where* $\% \in \{\wedge, \vee, \rightarrow\}$
(iii) $(A^i, B^j)^k \iff (A^{i+k}, B^{j+k})$

(3) *For any pair-sentence formula $(A^i, B^j) \in \boldsymbol{FOR_P}$ $(\exists i, j \in \mathbf{N})$, both values of stage number i and j are relative to each other because the interval of each stage number of formulas is absolute for the validity of each formula. So, we assume that: $(A^i, B^j) \rightarrow (A^i, B^j)^{\pm n}$*

for every $n \in \mathbf{N}$. Otherwise, if some sentence A has only a unit of pair-sentence formula, then we assume that: $A^i \to (A^i)^{\pm n}$ for every $i, n \in \mathbf{N}$.

The referential stage number will start from 0 and increase with depending on the referential frequency like $0, 1, 2, 3, \ldots$. If there exists a new sentence "A is B" for some sentences A, B on stage 0, then we assume a pair-sentence formula (A^0, B^1), which intends to show the referential relation between a sentence A on some stage (e.g., 0) and a sentence B on the next stage (e.g., 1). So, if we keep the interval of each stage number, then the referential relation also holds even if each stage number of formulas shift to another by the same value. On the other hand, if such a pair-sentence does not exist for some sentence A on stage 0, then the sentence A keeps its validity among on any stages.

Example 2.2 (1) *$\forall A^0 \in \boldsymbol{FOR_P}$, "$A^0$ is A^0" $\iff \exists (A^0)^0, (A^0)^1, ((A^0)^0, (A^0)^1) \in \boldsymbol{FOR_P}$ by Definition 2.1 (1). We have $(A^0)^0 \iff A^0$ and $(A^0)^1 \iff A^1$ by Definition 2.1 (2). So, we get {"A^0 is A^0"} $\iff \Gamma_1 = \{(A^0, A^1)\}$.*

(2) *Similarly, for any $A^0, B^0, C^0 \in \boldsymbol{FOR_P}$,*
(*i*) {"A^0 is not A^0"} $\iff \Gamma_2 = \{(A^0, \neg A^1)\}$
(*ii*) {"A^0 is not B^0", "B^0 is not C^0", "C^0 is A^0"}
$\iff \Gamma_3 = \{(A^0, \neg B^1), (B^0, \neg C^1), (C^0, A^1)\}$

In **SCI**, we can interpret "A^0 is A^0" as $A^0 \equiv A^0$ and "A^0 is not A^0" as $A^0 \equiv (\neg A)^0$. So, we could not deal with a Liar sentence in **SCI** since both of A and $\neg A$ are not identical on the same stage number 0. But it is possible in **PSC** because that A and its negation $\neg A$ are interpreted on the different stage numbers 0 and 1, respectively from Example 2.2.

Definition 2.3 *Let Γ be a set of pair-sentence formulas $\{(A^0, B_1^1), (B_1^0, B_2^1), (B_2^0, B_3^1), \ldots, (B_{n-1}^0, B_n^1)\}$ ($\exists n \in \mathbf{N}$). Then we get $\Gamma = \{(A^0, B_1^1), (B_1^1, B_2^2), (B_2^2, B_3^3), \ldots, (B_{n-1}^{n-1}, B_n^n)\}$ by Definition 2.1(3). So,*

(1) *We say that a sequence of formulas $A^0 B_1^1 B_2^2 \cdots B_n^n$ is a* referential pattern *of formula A generated from Γ.*

(2) *If A is belong to a set of formulas $\{B_1^1, B_2^2, \ldots, B_n^n\}$, we say that A has a* circular referential relation *with respect to Γ. Otherwise, A has a* non-circular referential relation *with respect to Γ.*

(3) *The* referential cycle number *of A with respect to Γ, $\tau(A, \Gamma)$ in symbol, is defined as follows:*
(*i*) $\tau(A, \Gamma) = 0$ *if* $A \notin \{B_1^1, B_2^2, \ldots, B_n^n\}$,
(*ii*) $\tau(A, \Gamma) = n$ *if* $A \in \{B_1^1, B_2^2, \ldots, B_n^n\}$ *and* $A = B_n^n$.
So, if A has a circular referential relation with respect to Γ, $\tau(A, \Gamma) \geq 1$. Otherwise, $\tau(A, \Gamma) = 0$.

(4) *If $\tau(A, \Gamma) \leq 1$, we say that A is* categorical *with respect to Γ. Otherwise, A is* paradoxical *with respect to Γ.*

Example 2.4 *We assume that the classical reasoning holds on each stage i and $(\neg\neg A \leftrightarrow A)^i$ implies $(\neg\neg A, A)^i$. Then we have:*

(1) *Let Γ_1 be $\{(A^0, A^1)\}$. Then we have $A^0 A^1 A^2 \ldots$ as a referential pattern of formula A generated from Γ_1. So, we get $\tau(A, \Gamma_1) = 1$.*

(2) *Let Γ_2 be $\{(A^0, \neg A^1)\}$. Then we have $A^0 (\neg A)^1 A^2 (\neg A)^3 \ldots$ as a referential pattern of formula A generated from Γ_2. So, we get $\tau(A, \Gamma_2) = 2$.*

(3) *Let* $\Gamma_3 = \{(A^0, \neg B^1), (B^0, \neg C^1), (C^0, A^1)\}$*. Then we have* $A^0(\neg B)^1 C^2 A^3 \ldots$ *as a referential pattern of formula* A *generated from* Γ_3*. So, we get* $\tau(A, \Gamma_3) = 3$*. Similarly, we get* $\tau(B, \Gamma_3) = 3$ *and* $\tau(C, \Gamma_3) = 3$*.*

Definition 2.5 (PSC system) *The axiomatic system* **PSC** *consists of two sets of schema* ***TFA*** *(truth functional axioms) and* ***PSA*** *(pair-sentence axioms) below. The only rule of inference is modus ponens:*

$(A1)$ $A \to (B \to A)$
$(A2)$ $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
$(A3)$ $A \wedge B \to A$
$(A4)$ $A \wedge B \to B$
$(A5)$ $A \to (B \to A \wedge B)$
$(A6)$ $A \to A \vee B$
$(A7)$ $B \to A \vee B$
$(A8)$ $(A \to C) \to ((B \to C) \to (A \vee B \to C))$
$(A9)$ $(A \to B) \to ((A \to \neg B) \to \neg A)$
$(A10)$ $\neg\neg A \to A$
$(E1)$ (A^0, A^0)
$(E2)$ $(A, B) \to (B, A)$
$(E3)$ $(A, B) \wedge (B, C) \to (A, C)$
$(C1)$ $(A, B) \to (\neg A, \neg B)$
(C2) $(A^i, B^j) \wedge (C^i, D^j)^k \to ((A^0 \wedge C^k)^i, (B^0 \wedge D^k)^j)$ $(\forall i, j, k \in \mathbf{N})$
(C3) $(A^i, B^j) \wedge (C^i, D^j)^k \to ((A^0 \vee C^k)^i, (B^0 \vee D^k)^j)$ $(\forall i, j, k \in \mathbf{N})$
(C4) $(A^i, B^j) \wedge (C^i, D^j)^k \to ((A^0 \to C^k)^i, (B^0 \to D^k)^j)$ $(\forall i, j, k \in \mathbf{N})$
(C5) $(A^i, B^j) \wedge (C^i, D^j)^k \to ((A^0, C^k)^i, (B^0, D^k)^j)$ $(\forall i, j, k \in \mathbf{N})$
$(P1)$ $(A, B) \to (A \to B)$
$(P2)$ $(A, B) \to (A, B)^{\pm n}$ $(\forall n \in \mathbf{N})$
$(P3)$ $A \to A^{\pm n}$ $(\forall n \in \mathbf{N})$ *if* A *has only a unit of pair-sentence formula*
(Mp) $\dfrac{A \quad A \to B}{B}$

The axioms in ***TFA*** with modus ponens as a single rule give an axiomatic system ***CL*** for the classical sentential logic. If we define a system $\mathbf{PSC}^0$ by restricting the stage number as $0 \in \mathbf{N}$ in a language $\mathcal{L}_P$, i.e., "A^0 is B^0" $\iff$ there exist A^0, B^0, $(A^0, B^0) \in \boldsymbol{FOR_P}^0$, and hence, eliminating axioms (P2) and (P3) from **PSC**. Then the system $\mathbf{PSC}^0$ is collapsed into systems **SCI** because that we can regard any pair-sentence formula $(A, B)^0$ as an identity formula $(A \equiv B)^0$ in **SCI** on stage 0.

Definition 2.6 (Derivability) *Let* Γ *be a finite set of pair-sentence formulas in a language* $\mathcal{L}_P$*,* X *a finite set of formulas,* A *a formula and* **PSC** *a system in* $\mathcal{L}_P$*. Then we say that:*

(1) A^j *is* derivable *from* X *based on* Γ *in* **PSC***,* $\mathbf{PSC}, X \vdash^{\Gamma} A^j$ *in symbol, if there is a sequence of formulas* $B_1^{i_1}, B_2^{i_2}, \ldots, B_{n-1}^{i_{n-1}}, B_n^{i_n} (n \geq 1)$ *such that* $B_n^{i_n} = A^j$ *and every formula in the sequence* $B_1^{i_1}, B_2^{i_2}, \ldots, B_{n-1}^{i_{n-1}}, A^j$ *is either an axiom of* **PSC***, or belongs to* $X \cup \Gamma$*, or is obtained by (Mp) rule from formulas occurring before it in the sequence.* n *is a* length of derivation A^j *from* X *based on* Γ *in* **PSC**.

(2) A *is* derivable *from* X *based on* Γ *in* **PSC***,* $\mathbf{PSC}, X \vdash^{\Gamma} A$ *in symbol, if there is a sequence of formulas* $B_1^0, B_2^0, \ldots, B_{n-1}^0, B_n^0 (n \geq 1)$ *such that* $B_n^0 = A^0$ *and every formula in the sequence* $B_1^0, B_2^0, \ldots, B_{n-1}^0,$ A^0 *is either an axiom of* **PSC***, or belongs to* $X \cup \Gamma$*, or is obtained by (Mp) rule from formulas occurring before it in the sequence.*

(3) *If* $X = \emptyset$*,* $\mathbf{PSC} \vdash^{\Gamma} A$ *in symbol,* A *is a* theorem *of* **PSC** *based on* Γ*.*

Example 2.7 *Let* $\Gamma_1 = \{(A^0, \neg A^1)\}$ *and* $\Gamma_2 = \{(A^0, \neg A^1), (A^0, A^3)\}$*. Then,*
(1) $\mathbf{PSC}, A^0 \vdash^{\Gamma_1} \neg A^1$

(2) $\mathbf{PSC} \vdash^{\Gamma_1} (A^0, \neg\neg A^2)$
(3) $\mathbf{PSC} \vdash^{\Gamma_2} \bot$

Proof. (1): 1. $(A^0, \neg A^1) \to (A^0 \to \neg A^1)$ (P1)
2. $A^0 \to \neg A^1$ (Γ_1,1,Mp)
3. A^0 (Hypothesis)
4. $\neg A^1$ (2,3,Mp)
(2): 1. $(A^0, \neg A^1) \to (A^1, \neg A^2)$ (P2)
2. $(A^1, \neg A^2)$ (Γ_1,1,Mp)
3. $(A^1, \neg A^2) \to (\neg A^1, \neg\neg A^2)$ (C1)
4. $(\neg A^1, \neg\neg A^2)$ (2,3,Mp)
5. $(A^0, \neg A^1) \wedge (\neg A^1, \neg\neg A^2) \to (A^0, \neg\neg A^2)$ (E3)
6. $(A^0, \neg\neg A^2)$ (Γ_1,4,5,Mp)
(3): 1. $[A^0]$ (Hypothesis)
2. $\quad \neg A^1$ (P1:$A^0 \wedge (A^0, \neg A^1) \to \neg A^1$)
3. $\quad\quad [\neg A^2]$ (Hypothesis)
4. $\quad\quad A^1$ (3,P1:$\neg A^2 \wedge (A^0, \neg A^1) \to A^1$)
5. $\quad\quad (\bot)^1$ (2,4)
6. $\quad\quad (\bot)^2$ (5,P3:$\bot^1 \to \bot^2$)
7. $\quad A^2$ (3,6)
8. $\quad \neg A^3$ (7,P2,P1:$A^2 \wedge (A^0, \neg A^1) \to \neg A^3$)
9. $\quad \neg A^0$ ($\Gamma_2 : \neg A^3 \to \neg A^0$)
10. $\neg A^0$ (1,9)
11. A^1 (10, C1, P1:$\neg A^0 \wedge (\neg A^0, \neg\neg A) \to \neg\neg A^1 \to A^1$)
12. $\neg A^2$ (11, P1:$A^1 \wedge (A^0, \neg A^1) \to \neg A^2$]
13. $\neg A^3$ (10,$\Gamma_2 : \neg A^0 \to \neg A^3$)
14. A^2 (13,P2,P1:$\neg A^3 \wedge (A^0, \neg A^1) \to A^2$)
15. $(\bot)^2$ (12,14)
16. $(\bot)^0$ (15,P3:$\bot^2 \to \bot^0$)
□

Lemma 2.8 *For a finite set of pair-sentence formulas Γ, any finite sets of formulas X, Y and any formulas A, B, C, we have the following: $\forall i, j, k \in \mathbf{N}$,*

(1) $\mathbf{PSC}, C^k \vdash^{\Gamma} C^k$ *holds.*

(2) $\mathbf{PSC}, X \vdash^{\Gamma} C^k$ *implies* $\mathbf{PSC}, A^i, X \vdash^{\Gamma} C^k$.

(3) $\mathbf{PSC}, A^i, A^i, X \vdash^{\Gamma} C^k$ *implies* $\mathbf{PSC}, A^i, X \vdash^{\Gamma} C^k$.

(4) $\mathbf{PSC}, X, A^i, B^j, Y \vdash^{\Gamma} C^k$ *implies* $\mathbf{PSC}, X, B^j, A^i, Y \vdash^{\Gamma} C^k$.

(5) $\mathbf{PSC}, X \vdash^{\Gamma} A^i$ *and* $\mathbf{PSC}, A^i, Y \vdash^{\Gamma} C^k$ *imply* $\mathbf{PSC}, X, Y \vdash^{\Gamma} C^k$.

Theorem 2.9 (Deduction Theorem) *For a finite set of pair-sentence formulas Γ, a finite set of formulas X and any formulas A, B,* $\mathbf{PSC}, X, A^i \vdash^{\Gamma} B^j$ *implies* $\mathbf{PSC}, X \vdash^{\Gamma} A^i \to B^j$ *for any $i, j \in \mathbf{N}$.*

Proof. Fix X and A^i and we prove by induction on the length k of derivation B^j from X and A^i based on Γ in $\mathbf{PSC}$. (i) Base step: We have to check the three cases: (case 1): B^j is one of axioms. Then we have the derivation using the axiom (A1): both $B^j \to (A^i \to B^j)$ and B^j imply $A^i \to B^j$. Hence we have $\mathbf{PSC}, X \vdash^{\Gamma} A^i \to B^j$. (case 2): B^j is one of $X \cup \Gamma$. This case is similar to the above. (case 3): B^j is just A^i. We have $A^i \to A^i$ as a theorem of $\mathbf{PSC}$ (see Proposition 2.13). (ii) Induction step: We have to check the four cases. But the first three cases are similar to the base step. (case 4): B^j is a result of derivation from $A_g^{i_g}$ and $A_h^{i_h}$ where

$g, h \le k$. Then $A_g^{i_g}$ is $A_h^{i_h} \to B^j$ or $(A_h^{i_h}, B^j)$. Because of $g, h \le k$, we can apply the induction hypothesis to $A_g^{i_g}$ and $A_h^{i_h}$. Hence we have the following:

$$\dfrac{\dfrac{\dfrac{\overset{(see\ Proposition\ 2.13)}{(A^i \to (A_h^{i_h}, B^j)) \to (((A_h^{i_h}, B^j) \to (A_h^{i_h} \to B^j)) \to (A^i \to (A_h^{i_h} \to B^j)))} \quad \overset{(I.Hypo)}{A^i \to (A_h^{i_h}, B^j)}}{((A_h^{i_h}, B^j) \to (A_h^{i_h} \to B^j)) \to (A^i \to (A_h^{i_h} \to B^j))}}{}$$

$$\dfrac{\overset{(see\ above)}{((A_h^{i_h}, B^j) \to (A_h^{i_h} \to B^j)) \to (A^i \to (A_h^{i_h} \to B^j))} \quad \overset{(P1)}{(A_h^{i_h}, B^j) \to (A_h^{i_h} \to B^j)}}{A^i \to (A_h^{i_h} \to B^j)}$$

$$\dfrac{\dfrac{\overset{(see\ Proposition\ 2.13)}{(A^i \to A_h^{i_h}) \to ((A^i \to (A_h^{i_h} \to B^j)) \to (A^i \to B^j))} \quad \overset{(I.Hypo)}{A^i \to A_h^{i_h}}}{(A^i \to (A_h^{i_h} \to B^j)) \to (A^i \to B^j)} \quad \overset{(see\ above\ or\ I.Hypo)}{A^i \to (A_h^{i_h} \to B^j)}}{A^i \to B^j}$$

So, $\mathbf{PSC}, X \vdash^{\Gamma} A^i \to B^j$ holds.

□

Definition 2.10 *Let Γ be a finite set of pair-sentence formulas and X a finite set of formulas in a language $\mathcal{L}_P$. Then we say that: (see [1])*

(1) X *is a* theory *if* $\{A; \mathbf{PSC}, X \vdash^{\Gamma} A\} = X$.

(2) X *is* consistent *if* $\{A; \mathbf{PSC}, X \vdash^{\Gamma} A\} \neq \boldsymbol{FOR_P}$.

(3) X *is* complete, (or maximal consistent) *if there does not exist a consistent set Y such that $X \subset Y$.*

Proposition 2.11 *For a finite set of pair-sentence formulas Γ, finite sets of formulas X, Y and any formula A, we have the following:*

(1) $\mathbf{PSC}, X \vdash^{\Gamma} A$ *if and only if* $\mathbf{PSC}, \{A; \mathbf{PSC}, X \vdash^{\Gamma} A\} \vdash^{\Gamma} A$.

(2) *For every consistent set X, there exists a complete set Y such that $X \subseteq Y$.*

(3) *For every complete set X, X is a theory.*

(4) *If* $\mathbf{PSC}, X \nvdash^{\Gamma} A$, *there is a complete set Y such that $X \subseteq Y$ and $A \notin Y$.*

Definition 2.12 (Elementary extensions of PSC) *Let us assume the following additional axioms:*

$(P4)$ $(A^i, B^j) \wedge (B \leftrightarrow C)^j \to (A^i, C^j)$ $(\forall i, j \in \mathbf{N})$
$(P5)$ $(A, A^{\pm n})$ $(\exists n \ge 1)$ $(n - reflexivity)$

Then, some elementary extensions of $\mathbf{PSC}$ *are defined as follows:*

(1) $\mathbf{PSC_B} \overset{\text{def}}{=} \mathbf{PSC} \cup \{(P4)\}$
(2) $\mathbf{PSC_n} \overset{\text{def}}{=} \mathbf{PSC} \cup \{(P5)\}$
(3) $\mathbf{PSC_{Bn}} \overset{\text{def}}{=} \mathbf{PSC} \cup \{(P4), (P5)\}$

Proposition 2.13 *For any $A, B \in \boldsymbol{FOR_P}$, the following are theorems of* $\mathbf{PSC}$ *based on* $\Gamma = \emptyset$:

(1) *The classical tautology formulas.*

(2) *The pair-sentence tautology formulas:*
(1°) $\mathbf{PSC} \vdash^{\emptyset} (A, B) \leftrightarrow (B, A)$
(2°) $\mathbf{PSC} \vdash^{\emptyset} (A, B) \to (A \leftrightarrow B)$
(3°) $\mathbf{PSC} \vdash^{\emptyset} \neg(A, \neg A)$
(4°) $\mathbf{PSC} \vdash^{\emptyset} (A, \top) \to A$

(5°) $\mathbf{PSC} \vdash^{\emptyset} (A, \bot) \to (\neg A)$
(6°) $\mathbf{PSC} \vdash^{\emptyset} ((A, B), \top) \to (A, B)$
(7°) $\mathbf{PSC} \vdash^{\emptyset} ((A, B), \bot) \to \neg(A, B)$
(8°) $\mathbf{PSC} \vdash^{\emptyset} ((A \to B), \top) \to (A \to B)$

Proof. (1): Suppose that there exist any formulas on stage $0 \in \mathbf{N}$. Then we can prove that every classical tautology formulas are also tautology in $\mathbf{PSC}$. For example, $\mathbf{PSC} \vdash^{\emptyset} A \to A$ is proved as follow:
1. $(A^0 \to ((A^0 \to A^0) \to A^0)) \to ((A^0 \to (A^0 \to A^0)) \to (A^0 \to A^0))$ (A2)
2. $(A^0 \to ((A^0 \to A^0) \to A^0))$ (A1)
3. $(A^0 \to (A^0 \to A^0)) \to (A^0 \to A^0)$ (1,2,Mp)
4. $A^0 \to (A^0 \to A^0)$ (A1)
5. $A^0 \to A^0$ (4,5,Mp)
6. $(A \to A)^0$ (Definition 2.1(2))

Moreover, $\mathbf{PSC} \vdash^{\emptyset} \neg A^i \to (A^i \to B^j)$ is proved as follow:
1. $[\neg A^i]$ (Hypothesis)
2. $[A^i]$ (Hypothesis)
3. $(\bot)^i$ (1,2)
4. $(\bot)^j$ (P3:$\bot^i \to \bot^j$)
5. B^j (CL:$\bot^j \to B^j$)
6. $A^i \to B^j$ (DT)
7. $\neg A^i \to (A^i \to B^j)$ (DT)

(2): Similar to the above. For example, (3°) is proved as follow:
(3°): 1. $(A^0, \neg A^0) \to (A^0 \to \neg A^0)$ (P1)
2. $\neg(A^0 \to \neg A^0) \to \neg(A^0, \neg A^0)$ (1,(1):$(A^0 \to B^1) \leftrightarrow (\neg B^1 \to \neg A^0)$)
3. $\neg(A^0, \neg A^0)$ ((1):$\neg(A^0 \to \neg A^0)$,2,Mp)
□

Proposition 2.14 *For any $A, B, C \in \boldsymbol{FOR_P}$, the following are theorems of* $\mathbf{PSC_B}$ *based on* $\Gamma = \emptyset$:

(1) *The classical tautology formulas.*

(2) *The pair-sentence tautology formulas additionally have the following:*
(1°) $\mathbf{PSC_B} \vdash^{\emptyset} (A, B) \leftrightarrow (\neg A, \neg B)$
(2°) $\mathbf{PSC_B} \vdash^{\emptyset} (A^i, B^j) \wedge (A \leftrightarrow C)^i \to (C^i, B^j)$ $(\forall i, j \in \mathbf{N})$
(3°) $\mathbf{PSC_B} \vdash^{\emptyset} (\neg\top, \bot)$
(4°) $\mathbf{PSC_B} \vdash^{\emptyset} ((A \to A), \top)$
(5°) $\mathbf{PSC_B} \vdash^{\emptyset} (A \vee \neg A, B \vee \neg B)$
(6°) $\mathbf{PSC_B} \vdash^{\emptyset} (A \wedge \neg A, B \wedge \neg B)$
(7°) $\mathbf{PSC_B} \vdash^{\emptyset} (\neg\neg A, A)$
(8°) $\mathbf{PSC_B} \vdash^{\emptyset} (\neg(A \vee B), \neg A \wedge \neg B)$
(9°) $\mathbf{PSC_B} \vdash^{\emptyset} (\neg(A \wedge B), \neg A \vee \neg B)$
(10°) $\mathbf{PSC_B} \vdash^{\emptyset} (A \wedge \top, A)$
(11°) $\mathbf{PSC_B} \vdash^{\emptyset} (A \vee \bot, A)$

Proof. (1),(2): The same way as Proposition 2.13. For example, (2°) and (4°) are proved as follow:
(2°): 1. $(A^i, B^j) \wedge (A \leftrightarrow C)^i$ (Hypothesis)
2. (A^i, B^j) (1,A3,Mp)
3. $(A \leftrightarrow C)^i$ (1,A4,Mp)
4. $(A^i, B^j) \to (B^j, A^i)$ (E2)
5. (B^j, A^i) (2,4,Mp)

6. $(B^j, A^i) \wedge (A \leftrightarrow C)^i$ (5,3,A5,Mp)
7. (B^j, C^i) (6,P4:$(B^j, A^i) \wedge (A \leftrightarrow C)^i \rightarrow (B^j, C^i)$,Mp)
8. $(B^j, C^i) \rightarrow (C^i, B^j)$ (E2)
9. (C^i, B^j) (7,8,Mp)
(4°): 1. $((A \rightarrow A)^0, (A \rightarrow A)^0)$ (E1)
2. $(A \rightarrow A)^0 \leftrightarrow (\neg A \vee A)^0$ (CL)
3. $((A \rightarrow A)^0, (A \rightarrow A)^0) \wedge ((A \rightarrow A) \leftrightarrow (\neg A \vee A))^0$ (1,2,A5,Mp)
4. $((A \rightarrow A)^0, (\neg A \vee A)^0)$ (1,2,3,P4,Mp)
5. $((A \rightarrow A), \top)^0$ ($\top := (A^0 \vee \neg A^0)$)

□

Proposition 2.15 *Let* $\Gamma_1 = \{(A^0, A^1)\}$ *be a set of pair-sentence formulas. Then for any* $A \in \boldsymbol{FOR_P}$ *and* $m, n \in \mathbf{N}$*, the following are theorems of* **PSC** (*also* $\mathbf{PSC_B}$) *based on* Γ_1 :

(1) *The classical tautology formulas additionally have the following:*
(1°) **PSC**(*also* $\mathbf{PSC_B}$) $\vdash^{\Gamma_1} A^m \vee \neg A^n$
(2°) **PSC**(*also* $\mathbf{PSC_B}$) $\vdash^{\Gamma_1} \neg(A^m \wedge \neg A^n)$
(3°) **PSC**(*also* $\mathbf{PSC_B}$) $\vdash^{\Gamma_1} A^m \leftrightarrow A^n$
(4°) **PSC**(*also* $\mathbf{PSC_B}$) $\vdash^{\Gamma_1} \neg(A^m \leftrightarrow \neg A^n)$

(2) *The pair-sentence tautology formulas additionally have the following:*
(1°) **PSC**(*also* $\mathbf{PSC_B}$) $\vdash^{\Gamma_1} (A^m, A^n)$
(2°) **PSC**(*also* $\mathbf{PSC_B}$) $\vdash^{\Gamma_1} \neg(A^m, \neg A^n)$

(3) $\mathbf{PSC_1}$(*also* $\mathbf{PSC_{B1}}$) $\vdash^{\emptyset} A$ *if and only if* **PSC**(*also* $\mathbf{PSC_B}$) $\vdash^{\Gamma_1} A$.

Proof. (1),(2): The same way as Proposition 2.13 (and also 2.14), and additionally we can prove the following:
1. (A^0, A^1) (Hypothesis of Γ_1)
2. $(A^0, A^1) \rightarrow (A^0, A^1)^1$ (P2)
3. (A^1, A^2) (1,2,Mp)
4. $(A^0, A^1) \wedge (A^1, A^2) \rightarrow (A^0, A^2)$ (E3)
5. $(A^0, A^1) \wedge (A^1, A^2)$ (1,3,A5,Mp)
6. (A^0, A^2) (4,5,Mp)
7. (A^0, A^l) (Similar to 1-6)
8. $(A^0, A^l) \rightarrow (A^0, A^l)^m$ (P2)
9. $\underline{(A^m, A^n)}$ (7,8,Mp, where $n = l + m$)
10. $(A^m, A^n) \rightarrow (A^m \leftrightarrow A^n)$ (Proposition 2.13)
11. $\underline{A^m \leftrightarrow A^n}$ (9,10,Mp)
12. $\underline{\neg(A^m \leftrightarrow \neg A^n)}$ (11,CL:$(A^0 \leftrightarrow B^1) \leftrightarrow \neg(A^0 \leftrightarrow \neg B^1)$)
13. $(A^m, \neg A^n) \rightarrow (A^m \leftrightarrow \neg A^n)$ (Proposition 2.13)
14. $\neg(A^m \leftrightarrow \neg A^n) \rightarrow \neg(A^m, \neg A^n)$ (13, CL:$(A^0 \rightarrow B^1) \leftrightarrow (\neg B^1 \rightarrow \neg A^0)$)
15. $\underline{\neg(A^m, \neg A^n)}$ (12,14,Mp)
16. $(A^m, A^n) \rightarrow (A^n, A^m)$ (E2)
17. $(A^n, A^m) \rightarrow (A^n \rightarrow A^m)$ (P1)
18. $A^n \rightarrow A^m$ (9,16,17,Mp)
19. $\neg A^n \vee A^m$ (18, CL:$(A^0 \rightarrow B^1) \leftrightarrow (\neg A^0 \vee B^1)$)
20. $\underline{A^m \vee \neg A^n}$ $(19, \mathrm{CL} : A^0 \vee B^1 \leftrightarrow B^1 \vee A^0)$
21. $\neg A^m \vee A^n \leftrightarrow \neg\neg(\neg A^m \vee A^n)$ $(20, \mathrm{CL} : \neg\neg A \leftrightarrow A)$
22. $\neg\neg(\neg A^m \vee A^n) \leftrightarrow \neg(\neg\neg A^m \wedge \neg A^n)$ $(21, \mathrm{CL} : \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B))$
23. $\underline{\neg(A^m \wedge \neg A^n)}$ $(20, 21, 22, \mathrm{CL} : \neg\neg A \leftrightarrow A, Mp)$
(3): $\mathbf{PSC_1}$ has $(A, A^{\pm 1})$ as an additional axiom. So, we have $(A^0, (A^0)^{\pm 1}) \Leftrightarrow (A^0, A^{\pm 1})$. This means equivalently to assume $\Gamma_1 = \{(A^0, A^1)\}$ in **PSC**.

□

Proposition 2.16 *Let $\Gamma_2 = \{(A^0, \neg A^1)\}$ be a set of pair-sentence formulas. Then for any $A \in \boldsymbol{FOR_P}$ and $l, m, n \in \mathbf{N}$, the following are theorems of* **PSC** *based on* Γ_2 :

(1) *The classical tautology formulas additionally have the following:*

(1°) $\mathbf{PSC} \vdash^{\Gamma_2} A^m \vee \overbrace{\neg \cdots \neg}^{l+1} A^{m+l}$, *and moreover,*
$\mathbf{PSC} \vdash^{\Gamma_2} A^m \vee \neg A^{m+2l'} \quad (l = 2l')$
$\mathbf{PSC} \vdash^{\Gamma_2} A^m \vee A^{m+2l'+1} \quad (l = 2l' + 1)$

(2°) $\mathbf{PSC} \vdash^{\Gamma_2} \neg(A^m \wedge \overbrace{\neg \cdots \neg}^{l+1} A^{m+l})$, *and moreover,*
$\mathbf{PSC} \vdash^{\Gamma_2} \neg(A^m \wedge \neg A^{m+2l'}) \quad (l = 2l')$
$\mathbf{PSC} \vdash^{\Gamma_2} \neg(A^m \wedge A^{m+2l'+1}) \quad (l = 2l' + 1)$

(3°) $\mathbf{PSC} \vdash^{\Gamma_2} A^m \leftrightarrow \overbrace{\neg \cdots \neg}^{l} A^{m+l}$, *and moreover,*
$\mathbf{PSC} \vdash^{\Gamma_2} A^m \leftrightarrow A^{m+2l'} \quad (l = 2l')$
$\mathbf{PSC} \vdash^{\Gamma_2} A^m \leftrightarrow \neg A^{m+2l'+1} \quad (l = 2l' + 1)$

(4°) $\mathbf{PSC} \vdash^{\Gamma_2} \neg(A^m \leftrightarrow \overbrace{\neg \cdots \neg}^{l+1} A^{m+l})$, *and moreover,*
$\mathbf{PSC} \vdash^{\Gamma_2} \neg(A^m \leftrightarrow \neg A^{m+2l'}) \quad (l = 2l')$
$\mathbf{PSC} \vdash^{\Gamma_2} \neg(A^m \leftrightarrow A^{m+2l'+1}) \quad (l = 2l' + 1)$

(2) *The pair-sentence tautology formulas additionally have the following:*

(1°) $\mathbf{PSC} \vdash^{\Gamma_2} (A^m, \overbrace{\neg \cdots \neg}^{l} A^{m+l})$

(2°) $\mathbf{PSC} \vdash^{\Gamma_2} \neg(A^m, \overbrace{\neg \cdots \neg}^{l+1} A^{m+l})$

Proof. (1),(2): The same way as Proposition 2.15.

□

Proposition 2.17 *Let $\Gamma_2 = \{(A^0, \neg A^1)\}$ be a set of pair-sentence formulas. Then for any $A \in \boldsymbol{FOR_P}$ and $l, m, n \in \mathbf{N}$, the following are theorems of* $\mathbf{PSC_B}$ *based on* Γ_2 :

(1) *The classical tautology formulas are the same as* **PSC** *based on* Γ_2.

(2) *The pair-sentence tautology formulas additionally have the following:*
(1°) $\mathbf{PSC_B} \vdash^{\Gamma_2} (A^m, A^{m+2l})$
(2°) $\mathbf{PSC_B} \vdash^{\Gamma_2} (A^m, \neg A^{m+2l+1})$
(3°) $\mathbf{PSC_B} \vdash^{\Gamma_2} \neg(A^m, \neg A^{m+2l})$
(4°) $\mathbf{PSC_B} \vdash^{\Gamma_2} \neg(A^m, A^{m+2l+1})$

(3) $\mathbf{PSC_{B2}} \vdash^{\emptyset} A$ *if and only if* $\mathbf{PSC_B} \vdash^{\Gamma_2} A$.

Proof. (1),(2): The same way as Proposition 2.14 and 2.15, and additionally if we notice that $(\overbrace{\neg \cdots \neg}^{2l'} A^{m+2l'} \leftrightarrow A^{m+2l'})$ and $(\overbrace{\neg \cdots \neg}^{2l'+1} A^{m+2l'+1} \leftrightarrow \neg A^{m+2l'+1})$, then we can prove the following:

(1°) $\mathbf{PSC} \vdash^{\Gamma_2} (A^m, \overbrace{\neg \cdots \neg}^{l} A^{m+l})$, hence,
$\mathbf{PSC_B} \vdash^{\Gamma_2} (A^m, A^{m+2l'}) \quad (l = 2l')$
$\mathbf{PSC_B} \vdash^{\Gamma_2} (A^m, \neg A^{m+2l'+1}) \quad (l = 2l' + 1)$

(2°) $\mathbf{PSC} \vdash^{\Gamma_2} \neg(A^m, \overbrace{\neg \cdots \neg}^{l+1} A^{m+l})$, hence,
$\mathbf{PSC_B} \vdash^{\Gamma_2} \neg(A^m, \neg A^{m+2l'}) \quad (l = 2l')$
$\mathbf{PSC_B} \vdash^{\Gamma_2} \neg(A^m, A^{m+2l'+1}) \quad (l = 2l' + 1)$

(3): $\mathbf{PSC_{B2}}$ has $(A, A^{\pm 2})$ as an additional axiom. So, we have $(A^0, (A^0)^{\pm 2}) \Leftrightarrow (A^0, A^{\pm 2})$. This means equivalently to assume $\Gamma_2 = \{(A^0, \neg A^1)\}$ in $\mathbf{PSC_B}$ because of (2):$\mathbf{PSC_B} \vdash^{\Gamma_2} (A^m, A^{m+2l})$.
$\square$

3 Semantices of PSC

Let us begin to consider the definition of semantices for **PSC** logic. We interpret $\mathcal{L}_P$ by using a classical truth assignment function $v : \boldsymbol{VAR_P} \to \{0,1\}$ where $\boldsymbol{VAR_P} = \bigcup_{i \in \mathbf{N}} \boldsymbol{VAR}^i$. Then we can easily extend this function v to the domain of all formulas in a language $\mathcal{L}_P$. The assignment for all logical connectives $\neg, \wedge, \vee, \to$ are as usual way, but we will use the truth transition function $\delta^{j-i} : \boldsymbol{TV}^i \to \boldsymbol{TV}^j$ to interpret a pair-sentence formula (A^i, B^j) where $\boldsymbol{TV}^i = \{v(A); A \in \boldsymbol{FOR_P}^i\}$ and $\boldsymbol{FOR_P}^i$ is a set of all formulas on stage i. The n-th order of truth transition function δ^n is defined as follow:

Definition 3.1 (Truth transition function) *Let Γ be a finite set of pair-sentence formulas and $v_0 \in \boldsymbol{TV}^0$ an initial truth value of assignment.*

(1) $\delta_\Gamma : \boldsymbol{TV}^0 \to \boldsymbol{TV}^1$ *is a* truth transition function *determined from* Γ.

(2) *Moreover, the following is a* sequence of truth transition functions *determined from* δ_Γ :
$${\delta_\Gamma}^0(v_0) = v_0$$
$${\delta_\Gamma}^{n+1}(v_0) = \delta_\Gamma({\delta_\Gamma}^n(v_0))$$
where $n \geq 0$ is an order of truth transition function.

(3) v_0 *is* n-reflexive *with respect to* Γ *if* ${\delta_\Gamma}^n(v_0) = v_0$ $(\exists n \in \mathbf{N})$.

(4) ${\delta_\Gamma}^{-1}$ *is a reverse truth transition function of* δ_Γ.

We notice that 1-reflexive assignments are fixed points of δ_Γ, 2-reflexive ones have 2 as a cycle number and every initial assignment v_0 is 0-reflexive. Then we can easily extend this function δ_Γ to the domain of all elements in an Boolean algebra as follows.

Definition 3.2 *Let Γ be a finite set of pair-sentence formulas, $\mathcal{A}_P = \langle \boldsymbol{A_P}, \sim, \cap, \cup, \supset, (\_ : \_), 1, 0 \rangle$ an* **PSC**-*algebra and $\boldsymbol{D_P}$ a subset of $\boldsymbol{A_P}$.*

(1) *An assignment of $\mathcal{A}_P$ is a homomorphism $v : \mathcal{L}_P \to \mathcal{A}_P$ such that the following hold: for any $A, B \in \boldsymbol{FOR_P}$,*
   (*i*) $v(A^i) \Longleftrightarrow (v(A))^i$ $(\forall i \in \mathbf{N})$
   (*ii*) $v(\neg A) \Longleftrightarrow \sim v(A)$
   (*iii*) $v(A \% B) \Longleftrightarrow v(A) \dot{\%} v(B)$ *where* $\% \in \{\wedge, \vee, \to\}$ *and* $\dot{\%} \in \{\cap, \cup, \supset\}$ *is an algebraic counterpart of* $\%$ *in order*
   (*iv*) $v((A, B)) \Longleftrightarrow (v(A) : v(B))$
   (*v*) $v(\top) = 1$ *and* $v(\bot) = 0$

(2) $\delta_\Gamma : {\mathcal{A}_P}^0 \to {\mathcal{A}_P}^1$ *is a Boolean transition function determined from Γ, where ${\mathcal{A}_P}^i$ is an Boolean algebra on order i $(i = 0, 1)$.*

(3) *The ordering of composed elements is the following: for every elements $a^m, b^n \in \boldsymbol{A_P}$ and number $l \in \mathbf{N}$,*
   (*i*) $(\sim a^m)^l \Longleftrightarrow \sim a^{m+l}$
   (*ii*) $(a^m \dot{\%} b^n)^l \Longleftrightarrow (a^{m+l} \dot{\%} b^{n+l})$ *where* $\dot{\%} \in \{\cap, \cup, \supset, :\}$

(4) (*i*) $\boldsymbol{D_P}$ *is* closed *if for every elements $a^m, b^n \in \boldsymbol{A_P}$, $a^m \in \boldsymbol{D_P}$ and $a^m \supset b^n \in \boldsymbol{D_P}$ imply $b^n \in \boldsymbol{D_P}$.* (*ii*) $\boldsymbol{D_P}$ *is* proper *if* $\boldsymbol{D_P} \neq \boldsymbol{A_P}$. (*iii*) $\boldsymbol{D_P}$ *is* admissible *if for every assignment v of $\mathcal{A}_P$ and formula $A \in \boldsymbol{TFA} \sqcup \boldsymbol{PSA}$, $v(A) \in \boldsymbol{D_P}$.* (*iv*) $\boldsymbol{D_P}$ *is* prime *if for every element $a^m \in \boldsymbol{A_P}$, $a^m \in \boldsymbol{D_P}$ or $\sim a^m \in \boldsymbol{D_P}$.* (*v*) $\boldsymbol{D_P}$ *is* transit *if for every elements $a^m, b^n \in \boldsymbol{A_P}$, $(a^m : b^n) \in \boldsymbol{D_P} \Longleftrightarrow {\delta_\Gamma}^{n-m}(a^m) = b^n$.* (*vi*) $\boldsymbol{D_P}$ *is* normal *if for every elements $a^m, b^m \in \boldsymbol{A_P}$, $(a^m : b^m) \in \boldsymbol{D_P} \Longleftrightarrow {\delta_\Gamma}^0(a^m) = b^m \Longleftrightarrow a^m = b^m$.*

(5) $\boldsymbol{D_P}$ *is* filter *if* $\boldsymbol{D_P}$ *is proper, closed and admissible.*

In the above definition, $\sim, \cap, \cup, \supset$ are as usual Boolean operators. If we assume a set of pair-sentence formulas as $\Gamma = \{(B^0_{i_1}, B^1_{j_1}), (B^0_{i_2}, B^1_{j_2}), \ldots, (B^0_{i_n}, B^1_{j_n})\}$ $(\exists n \in \mathbf{N})$, then we get the Boolean transition function δ_Γ as $\{\delta_\Gamma(v(B^0_{i_1})) = v(B^1_{j_1}), \delta_\Gamma(v(B^0_{i_2})) = v(B^1_{j_2}), \ldots, \delta_\Gamma(v(B^0_{i_n})) = v(B^1_{j_n})\}$.

Example 3.3 *Let* Γ_3 *be* $\{(A^0, \neg B^1), (B^0, \neg C^1), (C^0, A^1)\}$. *Then for any* $A^0, B^0, C^0 \in \boldsymbol{FOR_P}$ *there exist* $a^0, b^0, c^0 \in \boldsymbol{A_P}$ *such that* $v(A^0) = a^0, v(B^0) = b^0, v(C^0) = c^0$ *and* $\delta_{\Gamma_3} = \{\delta_{\Gamma_3}(a^0) = \sim b^1, \delta_{\Gamma_3}(b^0) = \sim c^1, \delta_{\Gamma_3}(c^0) = a^1\}$. *Moreover, we get the following sequence of Boolean transition functions:*

$\delta^0_{\Gamma_3}(a^0) = a^0$
$\delta^1_{\Gamma_3}(a^0) = \delta_{\Gamma_3}(\delta^0_{\Gamma_3}(a^0)) = \delta_{\Gamma_3}(a^0) = \sim b^1$
$\delta^2_{\Gamma_3}(a^0) = \delta_{\Gamma_3}(\delta^1_{\Gamma_3}(a^0)) = \delta_{\Gamma_3}(\sim b^1) = \sim \delta_{\Gamma_3}(b^1) = \sim\sim c^2$
$\delta^3_{\Gamma_3}(a^0) = \delta_{\Gamma_3}(\delta^2_{\Gamma_3}(a^0)) = \delta_{\Gamma_3}(\sim\sim c^2) = \sim\sim \delta_{\Gamma_3}(c^2) = \sim\sim a^3$ *and so on.*

Definition 3.4 *Let* Γ *be a finite set of pair-sentence formulas,* X *a finite set of formulas,* A *a formula and* $\mathcal{A}_P$ *an* **PSC**-*algebra.*

(1) $\mathcal{M}_P = \langle \mathcal{A}_P, \boldsymbol{D_P} \rangle$ *is a* **PSC**-*matrix if* $\boldsymbol{D_P}$ *is a filter in* $\mathcal{A}_P$.

(2) *Moreover,* $\mathcal{M}_P$ *is a* **PSC**-*model if* $\boldsymbol{D_P}$ *is a prime* ($1 \in \boldsymbol{D_P}$ *and* $0 \notin \boldsymbol{D_P}$), *transit filter.*

(3) A *is* true *in a* **PSC**-*model* $\mathcal{M}_P$ *under the assumption of* X *based on* Γ, $\mathcal{M}_P, X \models^\Gamma A$ *in symbol, if for every assigmnent* v *of* $\mathcal{A}_P$, $v(X \cup \Gamma) \subseteq \boldsymbol{D_P}$ *implies* $v(A) \in \boldsymbol{D_P}$.

(4) A *is* valid *under the assumption of* X *based on* Γ, $X \models^\Gamma A$ *in symbol, if for every* **PSC**-*model,* $\mathcal{M}_P, X \models^\Gamma A$.

Lemma 3.5 *Let* Γ *be a finite set of pair-sentence formulas and* $\delta_\Gamma : {\mathcal{A}_P}^0 \to {\mathcal{A}_P}^1$ *a Boolean transition function determined from* Γ, *where* ${\mathcal{A}_P}^i$ *is an Boolean algebra on order* i $(i = 0, 1)$. *Then we have:* $\forall a^m, b^n \in \boldsymbol{A_P}$, $\forall l \in \mathbf{N}$,

(1) $\sim {\delta_\Gamma}^l(a^m) = {\delta_\Gamma}^l(\sim a^m)$

(2) ${\delta_\Gamma}^l(a^m \dot{\%} b^n) = {\delta_\Gamma}^l(a^m) \dot{\%} {\delta_\Gamma}^l(b^n)$ *where* $\dot{\%} \in \{\cap, \cup, \supset, :\}$

(3) $\delta_\Gamma(a^m) = b^n \Longrightarrow \delta_\Gamma(a^{m+l}) = b^{n+l}$

Proof. By induction on the order length l of transition function ${\delta_\Gamma}^l$.
(1): Base step: Let $l = 0$. $\sim {\delta_\Gamma}^0(a^m) = \sim a^m = {\delta_\Gamma}^0(\sim a^m)$ by Definition 3.1.
Induction step: Assume that $\sim {\delta_\Gamma}^l(a^m) = {\delta_\Gamma}^l(\sim a^m)$ holds. Then,

$$
\begin{aligned}
\sim {\delta_\Gamma}^{l+1}(a^m) &= \sim \delta_\Gamma({\delta_\Gamma}^l(a^m)) && \text{(Definition 3.1)}\\
&= \delta_\Gamma(\sim {\delta_\Gamma}^l(a^m)) && \text{(I.H and l=1)}\\
&= \delta_\Gamma({\delta_\Gamma}^l(\sim a^m)) && \text{(I.H)}\\
&= {\delta_\Gamma}^{l+1}(\sim a^m) && \text{(Definition 3.1)}
\end{aligned}
$$

(2),(3): We can prove the similar way to (1).

□

Lemma 3.6 *For a finite set of pair-sentence formulas* Γ, *finite sets of formulas* X, Y, *a formula* A *and* $\mathcal{M}_P$ *a* **PSC**-*matrix, we have the following:*

(1) $\mathcal{M}_P, X \models^\Gamma A$ *for every* $A \in X \cup \Gamma$.

(2) $X \subseteq Y$ *and* $\mathcal{M}_P, X \models^\Gamma A$ *imply* $\mathcal{M}_P, Y \models^\Gamma A$.

(3) $\mathcal{M}_P, X \models^\Gamma A$ *if and only if* $\mathcal{M}_P, \{A; \mathcal{M}_P, X \models^\Gamma A\} \models^\Gamma A$.

Proposition 3.7 *Let Γ be a finite set of pair-sentence formulas, X a finite set of formulas, A a formula and $\mathcal{M}_P = \langle \mathcal{A}_P, \boldsymbol{D_P} \rangle$ a* **PSC***-model. Then* $\mathbf{PSC}, X \vdash^{\Gamma} A$ *implies* $\mathcal{M}_P, X \models^{\Gamma} A$ *for every* **PSC***-model* $\mathcal{M}_P$.

Proof. We prove by induction on the length k of derivation A from X based on Γ in **PSC**. (i) Base step: We have to check the two cases. (case 1): A is one of axioms. The classical truth functional axioms $\boldsymbol{TFA}$ are obvious and so we omitted. Assume that $v(A^0) = a^0, v(B^0) = b^0, v(C^0) = c^0, v(D^0) = d^0$, any $l, m, n \in \mathbf{N}$ and δ_Γ is a Boolean transition function determined from Γ. Then, (E1): $v((A^m, A^m)) = (a^m : a^m)$. For every δ_Γ, ${\delta_\Gamma}^0(a^m) = a^m$ by Definition 3.1. So $(a^m : a^m) \in \boldsymbol{D_P}$. (E2): $v((A, B) \to (B, A)) = (a^m : b^n) \supset (b^n : a^m)$. Assume $(a^m : b^n) \in \boldsymbol{D_P}$ $\Leftrightarrow {\delta_\Gamma}^{n-m}(a^m) = b^n$. So, we get ${\delta_\Gamma}^{-(n-m)}({\delta_\Gamma}^{n-m}(a^m)) = {\delta_\Gamma}^{-(n-m)}(b^n) \Leftrightarrow {\delta_\Gamma}^0(a^m) = {\delta_\Gamma}^{m-n}(b^n)$ $\Leftrightarrow {\delta_\Gamma}^{m-n}(b^n) = a^m \Leftrightarrow (b^n : a^m) \in \boldsymbol{D_P}$. (E3): $v((A, B) \wedge (B, C) \to (A, C)) = (a^l : b^m) \cap (b^m : c^n) \supset (a^l : c^n)$. Assume $(a^l : b^m) \cap (b^m : c^n) \in \boldsymbol{D_P} \Leftrightarrow ({\delta_\Gamma}^{m-l}(a^l) = b^m) \cap ({\delta_\Gamma}^{n-m}(b^m) = c^n)$. Then we get ${\delta_\Gamma}^{m-l}(a^l) = b^m \Rightarrow {\delta_\Gamma}^{n-m}({\delta_\Gamma}^{m-l}(a^l)) = {\delta_\Gamma}^{n-m}(b^m) = c^n \Leftrightarrow {\delta_\Gamma}^{n-l}(a^l) = c^n$. (C1): $v((A, B) \to (\neg A, \neg B)) = (a^m : b^n) \supset (\sim a^m : \sim b^n)$. Assume $(a^m : b^n) \in \boldsymbol{D_P} \Leftrightarrow {\delta_\Gamma}^{n-m}(a^m) = b^n$. Then we get ${\delta_\Gamma}^{n-m}(a^m) = b^n \Rightarrow \sim {\delta_\Gamma}^{n-m}(a^m) = \sim b^n \Leftrightarrow {\delta_\Gamma}^{n-m}(\sim a^m) = \sim b^n$ by Lemma 3.5 (1). Hence $(\sim a^m : \sim b^n) \in \boldsymbol{D_P}$. (C2)-(C5): By Lemma 3.5 (2). (P1) : $v((A, B) \to (A \to B)) = (a^m : b^n) \supset (a^m \supset b^n)$. Assume $(a^m : b^n) \in \boldsymbol{D_P} \Leftrightarrow {\delta_\Gamma}^{n-m}(a^m) = b^n$. So, we get $a^m \supset b^n$ by using a transition function ${\delta_\Gamma}^{n-m}$. (P2): By Lemma 3.5 (3). (P3): If A has only a unit of pair-sentence formula, then we have $(a^0 : a^0) \in \boldsymbol{D_P}$ and $(a^l : a^l) \in \boldsymbol{D_P}$ for any $l \in \mathbf{N}$ by (P2). So, there exists an identity tansition function id such that $id(a^m) = a^n$ for every $m, n \in \mathbf{N}$. (case 2): A is one of $X \cup \Gamma$. It is obvious from Lemma 3.6 (1). (ii) Induction step: We have to check the three cases. But the first two cases are similar to the base step. (case 3): A is a result of derivation from $A_g^{i_g}$ and $A_h^{i_h}$ where $g, h \le k$. Then $A_g^{i_g}$ is $A_h^{i_h} \to A$ or $(A_h^{i_h}, A)$. We have the two derivations: $\dfrac{(A_h^{i_h}, A) \;\; (A_h^{i_h}, A) \to (A_h^{i_h} \to A)}{A_h^{i_h} \to A}$ and $\dfrac{A_h^{i_h} \to A \;\; A_h^{i_h}}{A}$. Because of $g, h \le k$, we can apply the induction hypothesis to $A_g^{i_g}$ and $A_h^{i_h}$. Hence we have $v(A_h^{i_h} \to A) = a_h^{i_h} \supset a^m \in \boldsymbol{D_P}$ or $v((A_h^{i_h}, A)) = (a_h^{i_h} : a^m) \in \boldsymbol{D_P}$, and $v(A_h^{i_h}) = a_h^{i_h} \in \boldsymbol{D_P}$. Moreover, $(a_h^{i_h} : a^m) \in \boldsymbol{D_P}$ implies $a_h^{i_h} \supset a^m \in \boldsymbol{D_P}$ by (P1). So, both $a_h^{i_h} \supset a^m \in \boldsymbol{D_P}$ and $a_h^{i_h} \in \boldsymbol{D_P}$ imply $a^m \in \boldsymbol{D_P}$.

□

Definition 3.8 *Let $\mathcal{M}_P = \langle \mathcal{A}_P, \boldsymbol{D_P} \rangle$ and $\mathcal{M}_P{}' = \langle \mathcal{A}_P{}', \boldsymbol{D_P}{}' \rangle$ be* **PSC***-matrices. Then a function $h : \mathcal{A}_P \mapsto \mathcal{A}_P{}'$ is a* matrix homomorphism *from $\mathcal{M}_P$ into $\mathcal{M}_P{}'$ if h is an algebraic isomorphism from $\mathcal{A}_P$ into $\mathcal{A}_P{}'$ and $h^{-1}(\boldsymbol{D_P}{}') = \boldsymbol{D_P}$.*

Proposition 3.9 *Let Γ be a finite set of pair-sentence formulas and X a finite set of formulas. If h is a matrix homomorphism from $\mathcal{M}_P$ into $\mathcal{M}_P{}'$, and which maps $\mathcal{A}_P$ onto $\mathcal{A}_P{}'$, then $\mathcal{M}_P, X \models^{\Gamma} A$ if and only if $\mathcal{M}_P{}', X \models^{\Gamma} A$.*

Definition 3.10 *Let $\mathcal{M}_P = \langle \mathcal{A}_P, \boldsymbol{D_P} \rangle$ be a* **PSC***-matrix. Then we define the following:* $\forall a^m, b^n \in \boldsymbol{A_P}$,

(1) *$\approx$ is a binary relation on $\boldsymbol{A_P}$ such that $a^m \approx b^n \Leftrightarrow (a^m : b^n) \in \boldsymbol{D_P}$.*

(2) *$|a^m|$ is the congruence class of element a^m, i.e., $|a^m| = \{b^n; a^m \approx b^n\}$.*

(3) *$\boldsymbol{A_P}/\approx$ is the set of congruence classes of elements of $\boldsymbol{A_P}$, i.e.,* $\boldsymbol{A_P}/\approx = \{|a^m|; a^m \in \boldsymbol{A_P}\}$.

(4) *$\mathcal{A}_P/\approx = \langle \boldsymbol{A_P}/\approx, \sim, \cap, \cup, \supset, (\_ : \_), |1|, |0| \rangle$ is an* **PSC***-algebra with the following definitions: for every $|a^m|, |b^n| \in \boldsymbol{A_P}/\approx$,*
  (i) $\sim |a^m| \Longleftrightarrow | \sim a^m|$
  (ii) $|a^m| \; \dot{\%} \; |b^n| \Longleftrightarrow |a^m \; \dot{\%} \; b^n|$ *where* $\dot{\%} \in \{\cap, \cup, \supset, :\}$

Proposition 3.11 *Let Γ be a finite set of pair-sentence formulas, X a finite set of formulas, A a formula and $\mathcal{M}_P = \langle \mathcal{A}_P, \boldsymbol{D_P} \rangle$ a* **PSC***-matrix. Then we have the following:*

(1) *$\boldsymbol{D_P}/\approx$ is a filter in $\boldsymbol{A_P}/\approx$. So, $\mathcal{M}_P/\approx = \langle \mathcal{A}_P/\approx, \boldsymbol{D_P}/\approx \rangle$ is a* **PSC***-matrix.*

(2) *Moreover, $\boldsymbol{D_P}/\approx$ is a transit filter in $\boldsymbol{A_P}/\approx$.*

(3) *$\boldsymbol{D_P}/\approx$ is prime if and only if $\boldsymbol{D_P}$ is prime in $\boldsymbol{A_P}$.*

(4) *The mapping $a^m \mapsto |a^m|$ is a matrix homomorphism from $\mathcal{M}_P$ onto $\mathcal{M}_P/\approx$. So, $\mathcal{M}_P, X \models^\Gamma A$ if and only if $\mathcal{M}_P/\approx, X \models^\Gamma A$.*

Proof. (1): $\boldsymbol{D_P}/\approx = \boldsymbol{A_P}/\approx$ implies $\boldsymbol{D_P} = \boldsymbol{A_P}$. Hence $\boldsymbol{D_P} \neq \boldsymbol{A_P} \Rightarrow \boldsymbol{D_P}/\approx \neq \boldsymbol{A_P}/\approx$. So, if $\boldsymbol{D_P}$ is proper, then $\boldsymbol{D_P}/\approx$ is also proper. Since $\boldsymbol{D_P}$ is closed, $a^m, a^m \supset b^n \in \boldsymbol{D_P}$ implies $b^n \in \boldsymbol{D_P}$. So, $|a^m|, |a^m \supset b^n| \in \boldsymbol{D_P}/\approx$ implies $|b^n| \in \boldsymbol{D_P}/\approx$. Here $|a^m \supset b^n| \Leftrightarrow |a^m| \supset |b^n|$ by Definition 3.10 (4). So, $\boldsymbol{D_P}/\approx$ is also closed. Since $\boldsymbol{D_P}$ is admissible, for every axiom A in $\boldsymbol{TFA} \sqcup \boldsymbol{PSA}$ and every assignment v of $\boldsymbol{A_P}$, $v(A) \in \boldsymbol{D_P}$. So, we get $|v(A)| \in \boldsymbol{D_P}/\approx$ by Definition 3.10 (4). Hence $\boldsymbol{D_P}/\approx$ is also admissible. (2): For every $|a^m|, |b^n| \in \boldsymbol{A_P}/\approx$ and every Boolean transition function $\delta_\Gamma : {\mathcal{A}_P}^0 \to {\mathcal{A}_P}^1$ determined from Γ, there exists $\dot{\delta_\Gamma} : {\mathcal{A}_P}^0/\approx \to {\mathcal{A}_P}^1/\approx$ such that $(|a^m| : |b^n|) \in \boldsymbol{D_P}/\approx \iff \dot{\delta_\Gamma}^{n-m}(|a^m|) = |b^n| \iff \dot{\delta_\Gamma}^{n-m}(\{a_1^{i_1}, a_2^{i_2}, \ldots\}) = \{b_1^{i_1+(n-m)}, b_2^{i_2+(n-m)}, \ldots\}$, where ${\delta_\Gamma}^{n-m}(a_1^{i_1}) = b_1^{i_1+(n-m)}, {\delta_\Gamma}^{n-m}(a_2^{i_2}) = b_2^{i_2+(n-m)}, \ldots$ hold. So, $\boldsymbol{D_P}/\approx$ is transit. (3): Assume $\boldsymbol{D_P}$ is prime, i.e., for every $a^m \in \boldsymbol{A_P}$, $a^m \in \boldsymbol{D_P}$ or $\sim a^m \in \boldsymbol{D_P}$. So, $|a^m| \in \boldsymbol{D_P}/\approx$ or $|\sim a^m| \in \boldsymbol{D_P}/\approx \Leftrightarrow \sim |a^m| \in \boldsymbol{D_P}/\approx$. Hence $\boldsymbol{D_P}/\approx$ is also prime. The converse direction is also similar. (4): For every $a^m, b^n \in \boldsymbol{A_P}$, $a^m \not\approx b^n$, i.e., ${\delta_\Gamma}^{n-m}(a^m) \neq b^n$ implies $|a^m| \neq |b^n|$. Also for every $b \in \boldsymbol{A_P}/\approx$ there exists $a^m \in \boldsymbol{A_P}$ such that $|a^m| = b$. So, the mapping $a^m \mapsto |a^m|$ is both 1-1 and onto. Hence we get the result by Proposition 3.9.

□

Theorem 3.12 (Completeness) *Let Γ be a finite set of pair-sentence formulas, X a finite set of formulas, A a formula and $\mathcal{M}_P = \langle \mathcal{A}_P, \boldsymbol{D_P} \rangle$ a* **PSC***-model.*

(1) *X is consistent if and only if there exists a model $\mathcal{M}_P$ and an assignment v of $\mathcal{A}_P$ such that $X \subseteq v^{-1}(\boldsymbol{D_P})$.*

(2) $\mathbf{PSC}, X \vdash^\Gamma A$ *if and only if for every* **PSC***-model $\mathcal{M}_P$, $\mathcal{M}_P, X \models^\Gamma A$.*

(3) $\mathbf{PSC} \vdash^\Gamma A$ *if and only if for every* **PSC***-model $\mathcal{M}_P$, $\mathcal{M}_P \models^\Gamma A$.*

(4) $\mathbf{PSC} \vdash^\emptyset A$ *if and only if for every* **PSC***-model $\mathcal{M}_P$, $\mathcal{M}_P \models^\emptyset A$.*

Proof. (1): Assume X is consistent. Then there exists a complete set Y such that $X \subseteq Y$ and $Y = \{A; \mathbf{PSC}, Y \vdash^\Gamma A\}$ by Proposition 2.11 (2). Since Y is proper, closed and admissible, $\mathcal{M} = \langle \mathcal{L}_P, Y \rangle$ is a **PSC**-matrix. Hence there exists a assignment $v : \mathcal{L}_P \mapsto \mathcal{A}_P$ such that v is a matrix homomorphism from $\mathcal{M}$ into $\mathcal{M}_P$ and both 1-1 and onto. So, we get $X \subseteq v^{-1}(\boldsymbol{D_P})$ by Definition 3.8. (2): $\Rightarrow)$: Proposition 3.7. $\Leftarrow)$: Assume $\mathbf{PSC}, X \not\vdash^\Gamma A$. Then there exists a complete set Y such that $X \subseteq Y = \{A; \mathbf{PSC}, Y \vdash^\Gamma A\}$ and $A \notin Y$ by Proposition 2.11 (4). Here for every $A^m, B^n \in \boldsymbol{FOR_P}$, we define a binary relation $\sim$ on $\boldsymbol{A_P}$ as follows: $A^m \sim B^n \iff (A^m, B^n) \in Y$. Then $\mathcal{M}/\sim = \langle \mathcal{L}_P/\sim, Y/\sim \rangle$ is a Lindenbaum-Tarski quotient model and $\mathcal{M} \simeq \mathcal{M}/\sim$ holds. For every assignment $v : \mathcal{L}_P \mapsto \mathcal{A}_P$, we have $\mathcal{M} \simeq \mathcal{M}_P$ and $\mathcal{M}/\sim \simeq \mathcal{M}_P/\approx$. So, we have:

$$
\begin{aligned}
\mathbf{PSC}, X \not\vdash^\Gamma A &\Longrightarrow A \notin Y && \text{(Proposition 2.11 (4))}\\
&\iff \mathbf{PSC}, Y \not\vdash^\Gamma A && (Y = \{A; \mathbf{PSC}, Y \vdash^\Gamma A\})\\
&\iff \mathcal{M}, Y \not\models^\Gamma A && (1)\\
&\iff \mathcal{M}/\sim, Y \not\models^\Gamma A && (\mathcal{M} \simeq \mathcal{M}/\sim)\\
&\iff \mathcal{M}_P/\approx, Y \not\models^\Gamma A && (\mathcal{M}/\sim \simeq \mathcal{M}_P/\approx)\\
&\iff \mathcal{M}_P, Y \not\models^\Gamma A && (\mathcal{M}_P \simeq \mathcal{M}_P/\approx)
\end{aligned}
$$

$$\Longleftrightarrow \mathcal{M}_P, X \not\models^{\Gamma} A \qquad (X \subseteq Y)$$

Hence the reverse direction also holds. (3): Restrict to $X = \emptyset$ in the previous result. (4): Similarly, restrict to both $X = \emptyset$ and $\Gamma = \emptyset$.

□

4 Conclusion

In this paper we proposed a system that allows to deal with paradoxical sentences, like a Liar sentence: "A is not A", and presented both an axiomatic system **PSC** and an adequate **PSC**-matrix semantics for it. Our calculus has a pair-sentence (A^i, B^j) $(\exists i, j \in \mathbf{N})$ form to show the referential relation between two situations of sentence A on stage i and sentence B on stage j. The referential relation is similar to identity in **SCI**, but more general notion just as a mutual link relation between two sentences, so even that can be established between contradict sentences. If we restrict each stage number as $0 \in \mathbf{N}$ in **PSC**, then a pair-sentence form (A^0, B^0) is equivalent to an identity equation $A^0 \equiv B^0$ in **SCI**. In this sence, the **PSC** is a conservative extension of **SCI**. In **PSC**-matrix semantics, a pair-sentence (A^i, B^j) form can be interpreted as ${\delta_\Gamma}^{j-i}(a^i) = b^j$ using some Boolean transition function δ_Γ determined from Γ. So, if we restrict each stage number as $0 \in \mathbf{N}$, then we get ${\delta_\Gamma}^0(a^0) = a^0 = b^0$ as a semantical interpretation, which is identical to a **PSC**-matrix with normal filter, just same as an adequate semantics of **SCI** system. Now we consider the transition behaviour of three sets of pair-sentence formulas in Example 2.2, i.e., (1) $\Gamma_1 = \{(A^0, A^1)\}$, (2) $\Gamma_2 = \{(A^0, \neg A^1)\}$ and (3) $\Gamma_3 = \{(A^0, \neg B^1), (B^0, \neg C^1), (C^0, A^1)\}$. (1): For every assignment v

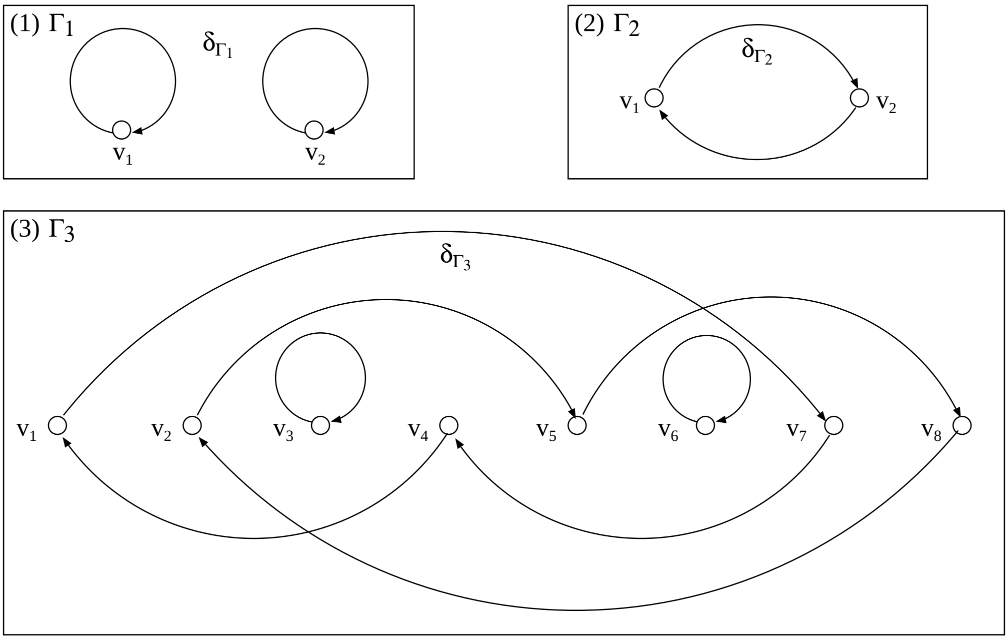

Figure 1: Boolean transition figures of $\delta_{\Gamma i}$ $(i \in \{1, 2, 3\})$

of $\mathcal{A}_P$, let $v_1 = v(A) = a$ and $v_2 = v(\neg A) = \sim a$. Then the behaviour of a Boolean transition function $\delta_{\Gamma_1}(a^0) = a^1$ shows the mapping of $v_1 \mapsto v_1$ and $v_2 \mapsto v_2$ at each transition. So, the assignment of senetnce A keeps constantly the same value among on each stage. Similarly, (2): the behaviour of $\delta_{\Gamma_2}(a^0) = \sim a^1$ shows the crossed mapping of $v_1 \mapsto v_2$ and $v_2 \mapsto v_1$ at each transition. So, the assignment of sentence A returns to an initial value at every after two transitions. (3): let $v_i = v(A) \times v(B) \times v(C)$ such that $v_1 = \langle a, b, c\rangle, v_2 = \langle a, b, \sim c\rangle, v_3 = \langle a, \sim b, c\rangle, v_4 = \langle a, \sim b, \sim c\rangle, v_5 = \langle \sim a, b, c\rangle, v_6 = \langle \sim a, b, \sim c\rangle, v_7 = \langle \sim a, \sim b, c\rangle$ and $v_8 = \langle \sim a, \sim b, \sim c\rangle$. Then the behaviour of δ_{Γ_3} shows the constant mapping on v_3, v_6 and three cyclic mapping on others. Moreover, we can consider more complicated set of pair-sentence formulas like $\Gamma_4 = \{(A^0, A^1), (B^0, C^1), (C^0, ((\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B))^1)\}$. In this case the behaviour of δ_{Γ_4} shows that if $v(A) = 1$, then four cyclic mapping on v_1, v_2, v_3, v_4 and otherwise, any transitions start from v_5, v_6, v_7, v_8 finally converge on the constant mapping on v_8.

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